Complex Analysis Solutions *

Final Semester 2014-2015

Problem 1

(i)
Let g be the primitive of f in $\mathbb{C}$. Then for any closed loop γ we have

$$\int_\gamma f(z)d(z) = \int_\gamma dg(z) = 0.$$

By choosing γ to be unit circle, we have the l.h.s. of to be $2\pi i \neq 0$. Therefore f does not have any primitive in $\mathbb{C}$.
(ii)
Because f is holomorphic on $B_1(0)$, f is also continuous on $B_1(0)$. Therefore $f(0) = \lim_{n\to\infty} f(\frac{1}{n}) = 0$. But we know that the zero set of any nontrivial holomorphic function is discrete. Therefore $f(z) = 0$ for any $z \in B_1(0)$.
(iii)
Because $|f(z)| = e^{|z|}$, f does not vanish in $B_1(0)$. The function $g = \frac{1}{f}$ is well defined and holomorphic. But from the given relation g attains maximum at $z = 0$, which contradicts the maximum modulus principle. Therefor the is no such f.

Problem 2

Applying the Schwarz's lemma to f, we have that $|f(z)| \leq |z|$. Consider the function f' on the set $B_{\frac{1}{2}}(0)$. From the Cauchy integral formula we get

$$f'(z_0) = \int\limits_{|z-z_0|=|z_0|} \frac{f(z)}{z-z_0}dz$$

for any $z_0 \in B_{\frac{1}{2}}(0)$. Therefore, using the fact that the maximum value of $|f|$ on the given contour is at most $2|z_0|$, we have $|f(z)| \leq 2|z|$ for any $z \in B_{\frac{1}{2}}(0)$. Define the function $g(z) = f'(\frac{z}{2})$. The function g satisfies the hypothesis of Schwarz's lemma. Therefore from Schwarz's lemma we have $|g'(0)| \leq 1$, which inturn gives the result $|f''(0)| \leq 2$.

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Problem 3

Define the function $g(z) = \frac{f(ez)}{e}$. The function $g : B_1(0) \to B_{\frac{1}{e}}(0)$ and every fixed point of g corresponds to a fixed point of f and vice-versa. We will show that g cannot have two fixed points. If so, let a and b be the two fixed points of g. Denote the map $\phi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$. Then the function $\phi_a \circ g \circ \phi_a^{-1}$ maps unit disk into unit disk and vanishes at 0. The point $\phi_a(b)$ is also a fixed point of $\phi_a \circ g \circ \phi_a^{-1}$. Therefore invoking Schwarz's lemma we obtain that the map $\phi_a \circ g \circ \phi_a^{-1}$ is an identity map, which in turn yields that g is identity. But g cannot be an identity map because the range of g is contained in $B_{\frac{1}{e}}(0)$. Hence there can only be atmost one fixed point of g.

Applying Rouche's theorem for the functions $g - Id$ and Id (Id- denotes identity function), we have $|g(z) - z - (-z)| = |g(z)| < |z|$, whenever $|z| > \frac{1}{e}$. Therefore $g - Id$ and Id have same number of zeros in $B_{\frac{1}{e}}(0)$. Hence g has exactly one fixed point.

Problem 4

Define the function $g = \frac{f}{r}$ which is holomorphic on $\mathbb{C} \setminus \{\alpha_n\}_{n\geq 1}$. Then it satisfies $|g| \leq 1$. Because g is bounded, we can extend g to the whole complex plane holomorphically (using Cauchy integral formula) and satisfying $|g| \leq 1$. Therefore g is bounded entire function and hence has to be constant c. Hence $f = cr$.

Problem 5

Consider the function $f : \{z : \Im(z) > 0\} \to B_1(0)$ defined by $f(z) = \frac{z-i}{z+i}$. This function is a bijection whose inverse is given by $f^{-1}(z) = -i\frac{w+1}{w-1}$. As the functions f and f^{-1} are rational and does not have poles in their respective domains, they are holomorphic. Therefore the upper half plane and unit disc are bi-holomorphically equivalent.

Problem 6

Consider the contour ($\gamma_n = \gamma_{1,n} \cup \gamma_{2,n}$) defined by the following two curves. $\gamma_{1,n} = \{z : -n \geq Re(z) \leq n; Im(z) = 0\}$ and $\gamma_{2,n} = \{z : |z| = n; Im(z) \geq 0\}$. For $n \geq 2$, invoking residue theorem, we have

$$\int_{\gamma_{n,1}} \frac{1}{1+z^2} d\gamma_{n,1}(z) + \int_{\gamma_{n,2}} \frac{1}{1+z^2} d\gamma_{n,2}(z) = \int_{\gamma_n} \frac{1}{1+z^2} d\gamma_n(z) = \frac{2\pi i}{i+i} = \pi. \tag{1}$$

$$\lim_{n\to\infty} |\int_{\gamma_{n,2}} \frac{1}{1+z^2} d\gamma_{n,2}(z)| \leq \lim_{n\to\infty} \int_{\gamma_{n,2}} |\frac{1}{(z-i)(z+i)}| d\gamma_{n,2}(z) \leq \lim_{n\to\infty} \frac{\pi n}{n^2-1} = 0 \tag{2}$$

Taking limits in (1) on both the sides and using (2) we have

$$\lim_{n\to\infty} \int_{\gamma_{n,1}} \frac{1}{1+z^2} d\gamma_{n,2}(z) = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi$$

Problem 7

Recall the fact that the image of any neighborhood of essential singularity, under the mapping of f, is dense in $\mathbb{C}$. Given that $Re(f) \geq 0$ on $P_1(0)$, therefore f cannot have essential singularity at 0. If f has a pole of order m at 0, then in a neighborhood of 0, f can be expanded as

$$f(z) = a_m z^{-m} + a_{m-1} z^{-(m-1)} + \dots.$$

We can choose z small enough such that $a_m z^{-m}$ is a purely negative real number and dominates all other terms together in modulus. For this choice of z, $Re(f(z)) < 0$ and contradicts the given assumption on f. Therefore f has a removable singularity at 0.